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UNIQUENESS AND LOCALIZATION—I. ASSOCIATIVE AND NON-ASSOCIATIVE ELASTOPLASTICITY

DAVIDE BIGONI† and TOMASZ HUECKEL
Department of Civil and Environmental Engineering, Duke University, Durham, NC 27706, U.S.A.

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Abstract—Localization of deformation into a planar band in the incremental response of elastoplastic material is studied in the case of small strains and rotations. The critical hardening modulus for localization is given in an explicit form (uncoupled from the band normal) for an arbitrary rate independent non-associative plasticity. Loss of uniqueness of the response is investigated in terms of positiveness of the second order work density. Criteria for loss of second order work positiveness and localization are compared for plane stress and plane strain. In these cases, for the associative flow rule, the threshold for the second order work positiveness coincides with the threshold for shear band formation. This coincidence may not, however, occur if localization into splitting mode is attained.

1. INTRODUCTION

A macroscopically homogeneous material element at a sufficiently low stress deforms in a homogeneous manner when a homogeneous stress is applied at its boundary. When the strain becomes larger it is inevitable that, due to actual irregularities in the distribution of mass or stiffness, etc., concentrations of high stress and/or strain occur. These concentrations may lead to various forms of local damage of material, like decohesion, faulting, nucleation of cavities or advanced slip of grains, depending much on the type of material. Forms, extent and interaction of these local singularities give rise to different micromechanisms of failure (Rice, 1976). Despite differences, the above local singularities have a common feature, which is the possibility of development into a macromechanism of failure. When occurring over a sufficiently large volume, this behavior can be modeled in terms of an elastoplastic continuum. Then such a macromechanism is perceived as a localization of strain over a more or less extended area. As a result, the deformation of the considered element ceases to be homogeneous.

A suitable tool for describing localization in terms of continuum theory is the strain rate discontinuity (Rudnicki and Rice, 1975; Rice and Rudnicki, 1980; Rice, 1976; Vardoulakis, 1976). The localization implies a non-uniqueness in the incremental elastoplastic response of a homogeneous, homogeneously strained body and, as shown by Rice (1976), also implies a vanishing speed of acceleration waves (Hadamard, 1903; Thomas, 1961; Hill, 1962; Mandel, 1966). This non-uniqueness consists of the possibility of the occurrence of more than one strain rate pattern related to the equilibrated fundamental stress state. In particular, a uniform stepwise strain rate field within a planar band superimposed on the homogeneous strain rate field appears to be admissible. From the phenomenological point of view, the band may be viewed as a macroscopic representation of the concentration of the above mentioned microstructural defects and/or microslips.

In general, the question of non-uniqueness of the elastoplastic incremental response should be addressed in the context of a boundary value problem, as formulated by Hill (1958, 1959, 1978) for associative flow-rule, or by Maier (1970), Hueckel and Maier (1977), Raniecki (1979) and Raniecki and Bruhns (1981) for non-associative flow rules. In this context, sufficient conditions for uniqueness were formulated. In principle, any sufficient condition for uniqueness is lost in the deformation process before the condition for a particular form of non-uniqueness, such as a localization, is met. Therefore, a possibility

† Visiting Research Associate from the University of Bologna, Bologna, Italy.

exists that other forms of non-uniqueness occur before the condition of localized strain is reached.

The question whether indeed the band formation can be preceded by other kinds of non-uniqueness is crucial from the point of view of constitutive modeling. In fact, it is believed that a constitutive equation can only be used when localization of deformation is excluded. After the localization is attained, the material behavior should be characterized in different terms, such as fracture propagation in rock and concrete or the force–displacement relationship at the shear band in soil (Rice, 1980; Ruina, 1980; Vardoulakis, 1981).

Until now an answer to the above question has been given for two classes of particular conditions. The first class is very restrictive and consists of a prescribed motion over all the boundary of a homogeneously deformed and stressed elastoplastic body with an associative flow rule. For this case, Hill (1962) has shown that the condition for vanishing speed of acceleration waves, and therefore for strain localization, does coincide with the sufficient condition for the loss of uniqueness of response which he previously gave (Hill, 1958; see also Rice, 1976).

The second class refers to plane strain tension and compression in the presence of large displacement gradients. For this case Hill and Hutchinson (1975) and Needleman (1979) analyzed the system of governing equations and found that while shear band localization may occur when the system is hyperbolic, various forms of diffuse bifurcation may precede the localization, when the system is still elliptic. For example, a diffuse necking is shown to occur during plane strain tension of an incompressible material block, well before the shear band forms.

The focus of the present work is on the loss of uniqueness and localization in materials exhibiting strain-softening and non-normality occurring at small strains and small rotations. These effects are typical for soils and granular materials as well as for concrete, masonry and rock.

Two particular criteria related to the uniqueness in the local elastoplastic response are discussed in detail (Section 2). These are loss of positiveness of second order work and strain localization into a narrow band. This choice is motivated by the fact that the sufficient criterion for uniqueness is never violated if the second order work density is positive (Raniecki, 1979; Raniecki and Bruhns, 1981). Moreover, the other possible criteria, i.e. loss of strong ellipticity and loss of positiveness of eigenvalues of the acoustic tensor (Mandel, 1966), occur between, or coincide with, loss of positiveness of second order work and localization.

In Section 3 a decoupled form of the condition for localization given by Rice (1976) has been obtained for any smooth yield function and non-associative law in which critical hardening modulus and band inclination are expressed explicitly. An uncoupled form has been available until now only for Drucker–Prager (Huber–von Mises as a particular case) yield surfaces (Rudnicki and Rice, 1975).

The formulation of the decoupled criterion for localization allows for a systematic analysis of localization modes and for comparison with the criterion for second order work for general unconstrained (3-D) and constrained (plane strain and plane stress) cases. Under plane strain and plane stress conditions, in the presence of the associative flow rule, the formation of shear bands, as opposed to splitting mode discontinuity, implies a loss of second order work positive definiteness (Bigoni and Hueckel, 1990a). For non-associative plasticity, such circumstances were not found in a general unconstrained case, nor in plane strain and plane stress (Section 4).

2. NON-UNIQUENESS THRESHOLDS IN ELASTOPLASTIC RATE PROBLEMS

A sufficient criterion for uniqueness of a generic boundary value problem, in which static and kinematic constraints are imposed on specific portions of the boundary, may be obtained from a straightforward application of the virtual work principle (Hill, 1958) in the form:

$$\int_V \Delta\dot{\sigma} : \Delta\dot{\varepsilon} \, dV > 0 \tag{1}$$

provided that the stress and strain rate fields $\Delta\dot{\sigma}$ and $\Delta\dot{\varepsilon}$ fulfil null traction rate and velocities conditions at appropriate portions of the boundary of the body of volume V.† Here, σ and ε are the second order tensors of stress and strain respectively, a dot denotes a rate and Δ refers to a difference between two alternative admissible fields.

The further analysis is restricted to elastic–plastic solids in which stress and strain rates are related through a fourth order tensor $\mathbf{D}$ in general non-symmetric:

$$\dot{\sigma} = \mathbf{D} : \dot{\varepsilon}. \tag{2}$$

Tensor $\mathbf{D}$ is different in loading and in unloading. Assuming that the elastic behavior (unloading) at every point is characterized by a positive definite tensor, the condition (1) is always fulfilled (Raniecki, 1979; Raniecki and Bruhns, 1981) if a much more restrictive condition holds, requiring that at every point of the body the second order work $\mathscr{L}$ is positive for every strain rate and corresponding stress rate (2):

$$\mathscr{L} = \tfrac{1}{2}\dot{\sigma} : \dot{\varepsilon} > 0. \tag{3}$$

Equation (3) will be referred to as the local criterion for uniqueness. The central property of the local criterion is the restriction it imposes on the constitutive law (2). In fact, substituting (2) into criterion (3), an equivalent condition of positive definiteness of the constitutive rate tensor $\mathbf{D}$ is obtained:

$$\mathbf{x} : \mathbf{D} : \mathbf{x} > 0, \quad \forall \mathbf{x} \in \mathrm{Sym} - \{0\}. \tag{4}$$

The above requirement restricts only the tensor $\mathbf{D}$ during elastoplastic loading.

A condition weaker with respect to the criterion (3), but sufficient to exclude strain localization, may be obtained by specializing tensor $\mathbf{x}$ in (4) to a particular rate deformation mode defined by a tensor product $\mathbf{g} \otimes \mathbf{n}$ of a versor $\mathbf{n}$ and a vector $\mathbf{g}$. This yields the requirement of the positive definiteness of every acoustic tensor $\mathbf{n} \cdot \mathbf{D} \cdot \mathbf{n}$, i.e.:

$$\mathbf{g} \otimes \mathbf{n} : \mathbf{D} : \mathbf{n} \otimes \mathbf{g} > 0, \quad \forall \mathbf{g} \neq 0 \; \forall \mathbf{n} \ni |\mathbf{n}| = 1. \tag{5}$$

The requirement (5) is the condition of strong ellipticity of the system of differential equations governing the local incremental equilibrium.

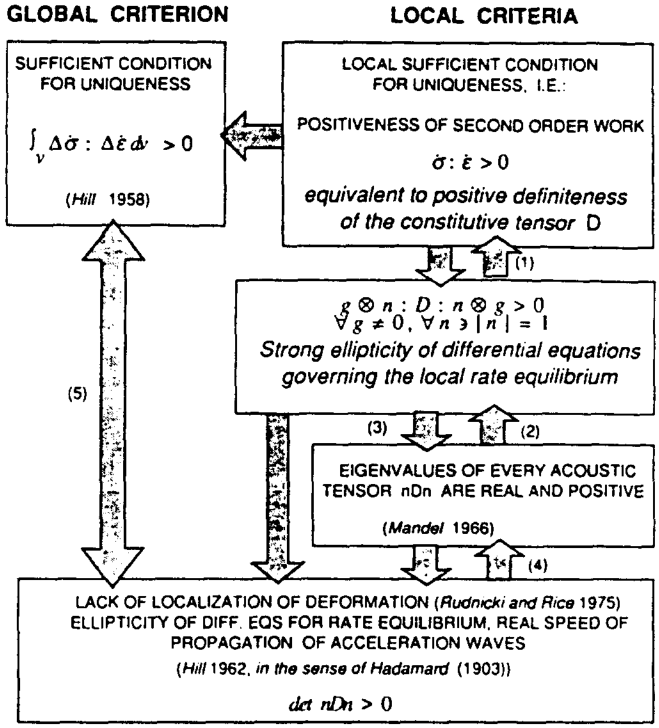
If the constitutive tensor $\mathbf{D}$ is symmetric, the condition (5) is equivalent to the requirement that all eigenvalues of the acoustic tensor are real and strictly positive. The latter condition was stated by Mandel (1966) as a threshold to material stability. A particular type of non-uniqueness in the form of the strain localization into a planar band (Rice, 1976; Rudnicki and Rice, 1975; Rice and Rudnicki, 1980; Vardoulakis, 1976) is attained when the system of differential equations governing the local rate equilibrium suffers a loss of ellipticity:

$$\det \mathbf{n} \cdot \mathbf{D} \cdot \mathbf{n} = 0 \tag{6}$$

where $\mathbf{n}$ is the versor normal to the planar band.

In the restricted case of plane strain isochoric motion for associative plasticity, all the above local criteria collapse into one (Prevost, 1987). The relationships between the criteria

† The criterion (1) may be obtained using the classical argument of Kirchhoff for the infinitesimal elastic theory (see, e.g. Gurtin, 1972, pp. 102–105).



(1) for plane isochoric motion.
(2) for associative plasticity.
(3) only if all eigenvalues of tensor nDn are real.
(4) for associative plasticity.
(5) for associative plasticity, constrained kinematics over the whole boundary of the body and homogeneous conditions before bifurcation (Rice 1976)

Fig. 1. Relationships between criteria for uniqueness, second order work, strong ellipticity, Mandel's stability and localization.

corresponding to the above discussed thresholds are represented in Fig. 1. While a double arrow in the figure denotes an identity, a single arrow means an implication. Numerical captions describe conditions under which the implications take place.

This paper investigates the relationship between the criteria for lack of localization, and for positiveness of second order work. The two criteria will be discussed in the context of non-associative plasticity. In non-associative plasticity the plastic strain rate tensor $\dot{\varepsilon}^p = \dot{\varepsilon} - \dot{\varepsilon}^e$ is expressed as:

$$\dot{\varepsilon}^p = \dot{\Lambda}\mathbf{P}, \tag{7}$$

where the second order tensor **P**, defining the mode of the plastic strain rate, is generally different, but assumed to be co-axial with the yield surface gradient **Q**. The assumption of co-axiality of tensors **P** and **Q**, which is essential for further derivations, is met in most of the known non-associative plasticity rules. The tensor $\dot{\varepsilon}^e$ represents the elastic strain rate and the scalar $\dot{\Lambda}$ is referred to as the plastic multiplier.

The stress rate may be related to the total strain rate $\dot{\varepsilon}$ as:

$$\dot{\sigma} = \mathbf{E} : \dot{\varepsilon} - \dot{\Lambda}\mathbf{E} : \mathbf{P} \tag{8}$$

subject to the conditions:

$$\dot{\Lambda} \geqslant 0, \quad \dot{f} \leqslant 0, \quad \dot{f}\dot{\Lambda} = 0. \tag{9}$$

In eqn (8), **E** is the fourth order isotropic elastic tensor. The yield surface $f(\sigma, k) = 0$ is a smooth function of stress and of the plastic isotropic hardening parameter k, depending on the plastic strain history.

The elastoplastic stiffness tensor $\mathbf{D}$, obtained from eqns (8) and (9), is:

$$\mathbf{D} = \mathbf{E} - \frac{(\mathbf{P}:\mathbf{E}) \otimes (\mathbf{Q}:\mathbf{E})}{H + \mathbf{P}:\mathbf{E}:\mathbf{Q}}, \text{ subject to (9).} \tag{10}$$

When the plastic hardening modulus H:

$$H = -\frac{\partial f}{\partial \boldsymbol{\varepsilon}^p}:\mathbf{P} \tag{11}$$

is positive, strain hardening occurs, while the material exhibits softening if H is negative. When H is zero the case of perfect plasticity is recovered. Softening is assumed not to exceed the snap-back threshold, i.e. the hardening modulus must be such that $H > -\mathbf{P}:\mathbf{E}:\mathbf{Q}$. A value of $H = H'_{cr}$ denotes (see, e.g., Rudnicki and Rice, 1975) the threshold between processes for which the localization is excluded ($H > H'_{cr}$) for every deformation rate and those for which it is not ($H \leqslant H'_{cr}$). An analogous threshold $H = H''_{cr}$ is used to delimit between processes ($H > H''_{cr}$) for which the uniqueness of response is ensured for every deformation rate field and those for which it is not ($H \leqslant H''_{cr}$) (Mróz, 1963).

Following the definitions (3) and (4) and using the constitutive relationships (7)–(11), it was shown that the threshold at which the loss of positiveness of the second order work occurs is determined by the critical plastic hardening modulus (Maier and Hueckel, 1979; Hueckel and Maier, 1977):

$$H''_{cr} = \tfrac{1}{2}[\sqrt{(\mathbf{P}:\mathbf{E}:\mathbf{P})(\mathbf{Q}:\mathbf{E}:\mathbf{Q})} - \mathbf{P}:\mathbf{E}:\mathbf{Q}]. \tag{12}$$

For associative plasticity ($\mathbf{P} = \mathbf{Q}$) the critical hardening modulus (12) becomes zero. The deformation rate at which second order work reduces to zero is:

$$\dot{\boldsymbol{\varepsilon}}^0 = \alpha(\sqrt{\mathbf{P}:\mathbf{E}:\mathbf{P}}\,\mathbf{Q} + \sqrt{\mathbf{Q}:\mathbf{E}:\mathbf{Q}}\,\mathbf{P}), \quad \forall \alpha \in \mathbb{R} - \{0\}. \tag{13}$$

The localization of deformation into planar bands takes place when a strain rate discontinuity occurs across the plane of the band. In order to be kinematically admissible, the strain rate discontinuity must satisfy the Maxwell compatibility conditions (see, e.g., Thomas, 1961):

$$[\![\dot{\boldsymbol{\varepsilon}}]\!] = \tfrac{1}{2}(\mathbf{g} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{g}) \tag{14}$$

where $[\![\ ]\!]$ indicates the discontinuity, $\mathbf{n}$ is the unit vector normal to the band and $\mathbf{g}$ is the vector which defines the discontinuity in the velocity derivative.

The principal components of the strain rate jump tensor (14) can be determined once vectors $\mathbf{n}$ and $\mathbf{g}$ are found, if $\dot{\varepsilon}_1 \geqslant \dot{\varepsilon}_2 \geqslant \dot{\varepsilon}_3$:

$$\begin{aligned} [\![\dot{\varepsilon}_1]\!] &= (|\mathbf{g}| + \mathbf{g}\cdot\mathbf{n})/2 \\ [\![\dot{\varepsilon}_2]\!] &= 0 \\ [\![\dot{\varepsilon}_3]\!] &= (-|\mathbf{g}| + \mathbf{g}\cdot\mathbf{n})/2. \end{aligned} \tag{15a–c}$$

Depending on the directions of vectors $\mathbf{n}$ and $\mathbf{g}$, two principal types of strain rate discontinuity occur, which are referred to as split mode and shear band. Split mode takes place in the case of co-axiality of vectors $\mathbf{g}$ and $\mathbf{n}$. This includes both tensile and compressive strain rate across the band. Shear band takes place in the case where $\mathbf{g}$ and $\mathbf{n}$ are not parallel. The hardening modulus at which localization takes place in the given direction, specified by the versor $\mathbf{n}$ orthogonal to the band, is (Rice, 1976):

$$H'(\mathbf{n}) = 2G\Big[2\mathbf{n}\cdot\mathbf{P}\cdot\mathbf{Q}\cdot\mathbf{n}-(\mathbf{n}\cdot\mathbf{P}\cdot\mathbf{n})(\mathbf{n}\cdot\mathbf{Q}\cdot\mathbf{n})-\mathbf{P}:\mathbf{Q} - \frac{\nu}{1-\nu}(\mathbf{n}\cdot\mathbf{P}\cdot\mathbf{n}-\mathrm{tr}\,\mathbf{P})(\mathbf{n}\cdot\mathbf{Q}\cdot\mathbf{n}-\mathrm{tr}\,\mathbf{Q})\Big] \quad (16)$$

where G is the elastic shear modulus, ν is Poisson's ratio and tr indicates the trace of a tensor.

The velocity discontinuity vector $\mathbf{g}$, corresponding to a given $\mathbf{n}$, is a right eigenvector of the acoustic tensor $\mathbf{n}\cdot\mathbf{D}\cdot\mathbf{n}$, corresponding to a null eigenvalue:

$$\mathbf{g} = 2\mathbf{n}\cdot\mathbf{P}-\frac{1}{1-\nu}(\mathbf{n}\cdot\mathbf{P}\cdot\mathbf{n})\mathbf{n}+\frac{\nu}{1-\nu}(\mathrm{tr}\,\mathbf{P})\mathbf{n}. \quad (17)$$

To obtain the critical hardening modulus H'_{cr}, a constrained maximization of (16) over all possible directions $\mathbf{n}$ must be performed:

$$H'_{cr} = \max_{\mathbf{n}} H'(\mathbf{n}), \text{ subject to } |\mathbf{n}| = 1. \quad (18)$$

It may be observed that the expression for the critical hardening modulus (18) cannot be given explicitly in an arbitrary reference system, i.e. H'_{cr} is coupled with the vector $\mathbf{n}$. Numerical procedures were developed (see, e.g., Ortiz *et al.*, 1987) to arrive at the value of the critical hardening modulus.

In the following section it will be shown that, if a suitable reference system is employed, the critical modulus and the vector $\mathbf{n}$ may be reduced to an explicit, i.e. decoupled, form.

3. LOCALIZATION CRITERIA FOR 3-D, PLANE STRAIN AND PLANE STRESS CASES

An explicit expression for the critical hardening modulus will first be derived for the general case, i.e. unconstrained, kinematics. Special forms will be subsequently deduced for constrained cases, namely those of plane strain and plane stress. Analogously, in the next section, plane strain and plane stress specializations will be obtained for the criterion of second order work.

Three-dimensional case

First, the constrained maximization problem (18) is reduced to the unconstrained maximization of the Lagrangian function:

$$L(\mathbf{n},\beta) = 2G\Big[2\mathbf{n}\cdot\mathbf{P}\cdot\mathbf{Q}\cdot\mathbf{n}-(\mathbf{n}\cdot\mathbf{P}\cdot\mathbf{n})(\mathbf{n}\cdot\mathbf{Q}\cdot\mathbf{n})-\mathbf{P}:\mathbf{Q} - \frac{\nu}{1-\nu}(\mathbf{n}\cdot\mathbf{P}\cdot\mathbf{n}-\mathrm{tr}\,\mathbf{P})(\mathbf{n}\cdot\mathbf{Q}\cdot\mathbf{n}-\mathrm{tr}\,\mathbf{Q})\Big]-\beta(\mathbf{n}\cdot\mathbf{n}-1) \quad (19)$$

where β is a Lagrangian multiplier.

The extrema of (16) are thus characterized by the conditions:

$$\frac{\partial L}{\partial \mathbf{n}} = 0 \quad \text{and} \quad \frac{\partial L}{\partial \beta} = 0. \quad (20)$$

The conditions yield:

$$2\mathbf{n}\cdot\mathbf{P}\cdot\mathbf{Q}-\frac{1}{1-\nu}[(\mathbf{n}\cdot\mathbf{Q}\cdot\mathbf{n})\mathbf{n}\cdot\mathbf{P}+(\mathbf{n}\cdot\mathbf{P}\cdot\mathbf{n})\mathbf{n}\cdot\mathbf{Q}]+\frac{\nu}{1-\nu}[(\operatorname{tr}\mathbf{Q})\mathbf{n}\cdot\mathbf{P}+(\operatorname{tr}\mathbf{P})\mathbf{n}\cdot\mathbf{Q}]=\frac{\beta}{2G}\mathbf{n}$$

$$\mathbf{n}\cdot\mathbf{n}=1. \tag{21}$$

Choosing now the principal axes of stress (denoted by i, j, k) as the reference system, eqns (21) may be written explicitly in the form:

$$2n_iP_iQ_i-\frac{1}{1-\nu}[(\mathbf{n}\cdot\mathbf{Q}\cdot\mathbf{n})P_i+(\mathbf{n}\cdot\mathbf{P}\cdot\mathbf{n})Q_i]n_i+\frac{\nu}{1-\nu}[(\operatorname{tr}\mathbf{Q})P_i+(\operatorname{tr}\mathbf{P})Q_i]n_i=\frac{\beta}{2G}n_i$$

$$(i=1,2,3 \text{ not assumed})$$

$$\mathbf{n}\cdot\mathbf{n}=1 \tag{22}$$

where the index i refers to the components in the reference system of principal stresses. The solutions of (22) yield all extrema of (16). The extrema correspond to different inclinations of the band, characterized by versor **n** as follows:

(i) none of the components of the versor **n** is null;
(ii) one of the components of the versor **n** is null;
(iii) two components of the versor **n** are null.

In order to find the maximum (18) over all extrema, solutions of (22) corresponding to the above cases have to be found and compared. Let us now specify all extrema for the cases (i)–(iii).

(i) *None of the components of the versor* **n** *is null*. System (22) admits in this case a unique solution if and only if:

$$\Delta=-[(Q_1-Q_3)(P_2-P_3)-(Q_2-Q_3)(P_1-P_3)]^2\neq 0 \tag{23}$$

where indices 1–3 indicate principal components. Note that the determinant Δ always vanishes in the case of associative plasticity and for a special non-associative flow rule of the type (Bigoni and Hueckel, 1990b):

$$\mathbf{P}=\mathbf{Q}+\xi\mathbf{I} \tag{24}$$

in which ξ is any scalar function and **I** is the identity tensor.

The flow rules of type (24) are used in the description of zero-dilatancy pressure sensitive materials, e.g. the Jenike–Shield (1959) flow rule, in which the Drucker–Prager yield function and Huber–von Mises plastic potential are employed.

The three following sub-cases need to be examined for case (i):

(ia) $\Delta\neq 0$;
(ib) $\Delta=0$, $Q_i=Q_j$ and $P_i=P_j$;
(ic) $\Delta=0$, $Q_i=Q_j=Q_k$ and $P_i=P_j=P_k$.

Any other combinations of Δ and components of tensors **P** and **Q** correspond to impossible solutions of system (22). The following solutions are found for the above mentioned sub-cases:

(ia) $\Delta\neq 0$. System (22) admits a unique solution, for the three components of the versor **n** (Bigoni, 1991):

$$n_1^2 = \frac{1}{\Delta}\{2a(P_2-P_3)(Q_2-Q_3)-b[(P_1-P_3)(Q_2-Q_3)+(P_2-P_3)(Q_1-Q_3)]\}$$

$$n_2^2 = \frac{1}{\Delta}\{2b(P_1-P_3)(Q_1-Q_3)-a[(P_1-P_3)(Q_2-Q_3)+(P_2-P_3)(Q_1-Q_3)]\}$$

$$n_3^2 = 1-n_1^2-n_2^2 \tag{25}$$

where:

$$a = Q_1(P_1-P_3)+P_1(Q_1-Q_3)+\nu[Q_2(P_1-P_3)+P_2(Q_1-Q_3)] \tag{26}$$

$$b = Q_2(P_2-P_3)+P_2(Q_2-Q_3)+\nu[Q_1(P_2-P_3)+P_1(Q_2-Q_3)]. \tag{27}$$

To be an admissible solution, the obtained values of n_1^2, n_2^2 and n_3^2 must be internal to the inverval [0, 1]. If this condition is not met, the extrema of hardening modulus are to be searched in the solutions defined by cases (ii) and (iii) only. The hardening modulus, corresponding to the versor with the components defined in (25), can now be easily obtained by substitution into eqn (16).

(ib) $\Delta = 0$, $Q_i = Q_j$ *and* $P_i = P_j$. Tensors **Q** and **P** are symmetric with respect to axis k. Then the only component of **n** which may be determined from (22) is that along the symmetry axis k, i.e. versor **n** spans a cone with axis k. The components along i and j are not determined. Thus system (22) now admits ∞^1 solutions. For any pair n_i and n_j the same extremum value of (16) is obtained. The extremum (16) and the corresponding component n_k may be found, assuming one of the components of versor **n** along the i or j axes is zero (case ii). If the obtained extremum is found to be maximal over all the extrema defined by (22), the number of possible localization bands becomes infinite.

(ic) $\Delta = 0$, $Q_i = Q_j = Q_k$ *and* $P_i = P_j = P_k$. Tensors **Q** and **P** are symmetric with respect to all principal axes. The inclination of the localization band is indeterminate. System (22) now admits ∞^2 solutions. The critical hardening modulus may be found, without loss of generality, assuming two of the components of versor **n** to be zero. From (17) it can be seen that the localization mode corresponds to a splitting discontinuity.

(ii) *One of the components of the versor* **n** *is null.* The band normal lies in one of the planes formed by two of the principal axes of stress.

Suppose that $n_k = 0$. Analyzing the determinant of (22) it may be seen that in this case solutions are possible in the following two sub-cases only:

(iia) $Q_i \neq Q_j$ and $P_i \neq P_j$;
(iib) $Q_i = Q_j$ and $P_i = P_j$.

The following solutions are found for the above mentioned sub-cases.

(iia) $Q_i \neq Q_j$ *and* $P_i \neq P_j$. Tensors **Q** and **P** are not symmetric with respect to the axis k. System (22) is reduced to two equations for the two unknown components of versor **n**, which become:

$$n_k = 0, \quad n_j^2 = 1-n_i^2,$$

$$n_i^2 = (1-\nu)\frac{P_iQ_i-P_jQ_j}{(P_i-P_j)(Q_i-Q_j)} - \frac{Q_j}{2(Q_i-Q_j)} - \frac{P_j}{2(P_i-P_j)} + \frac{\nu(Q_i+Q_j+Q_k)}{2(Q_i-Q_j)} + \frac{\nu(P_i+P_j+P_k)}{2(P_i-P_j)}$$

$$(i, j, k \text{ not summed}). \tag{28}$$

The value of the corresponding extremum of the hardening modulus is found from (16) and the known components of vector **n**:

$$H'/(2G) = n_i^2\left\{2(P_iQ_i - P_jQ_j) - \frac{1}{1-\nu}[(P_i-P_j)Q_i + (Q_i-Q_j)P_j] + \frac{\nu}{1-\nu}[\mathrm{tr}\,\mathbf{Q}(P_i-P_j)\right.$$

$$\left.+\,\mathrm{tr}\,\mathbf{P}(Q_i-Q_j)]\right\} - n_i^4\frac{1}{1-\nu}(P_i-P_j)(Q_i-Q_j) - P_jQ_j - P_kQ_k - \frac{\nu}{1-\nu}(Q_j+Q_k)(P_j+P_k)$$

$$(i,\ j,\ k \text{ not summed}). \tag{29}$$

Note that, in order to yield an admissible solution, the components of tensors **Q** and **P** in eqn (28) must be such as to give a value of n_i^2 internal to the interval [0, 1]. When this condition is not met, the components (28) and the corresponding value of the modulus (29) are inadmissible. Thus a non-analytical extremum must be searched, corresponding to case (iii).

Vector **g**, corresponding to the components (28), is given by the expressions:

$$g_i = \left\{2P_i - \frac{1}{1-\nu}[n_i^2(P_i-P_j)+P_j] + \frac{\nu}{1-\nu}\mathrm{tr}\,\mathbf{P}\right\}n_i$$

$$g_j = \left\{2P_j - \frac{1}{1-\nu}[n_i^2(P_i-P_j)+P_j] + \frac{\nu}{1-\nu}\mathrm{tr}\,\mathbf{P}\right\}n_j$$

$$g_k = 0. \tag{30}$$

In order to obtain all the extrema of the hardening modulus, the index k has to be permuted between 1 and 3.

(iib) $Q_i - Q_j$ *and* $P_i = P_j$. Tensors **Q** and **P** are symmetric with respect to axis k. The components along i and j are not determined. Thus system (22) now admits ∞^1 solutions. For any pair n_i and n_j the same extremum value of (16) is obtained. The extremum (16) may be found assuming one of the components of versor **n** along the i or j axes is zero (case iii). If the obtained extremum is found to be maximal over all the extrema defined by (22), the number of possible localization bands becomes infinite. The localization mode corresponds to a splitting discontinuity.

(iii) *Two components of the versor* **n** *are null.* The band normal is orthogonal to two of the principal axes of stress. Suppose that $n_i = 1$, $n_j = n_k = 0$, the extrema of the hardening modulus are given by the expression:

$$H' = -\frac{2G}{1-\nu}[P_kQ_k + P_jQ_j + \nu(P_kQ_j + P_jQ_k)] \tag{31}$$

where the indices, not summed, are to be permuted between 1 and 3 in order to obtain all the extrema of (16). Vector **g** has the components:

$$g_i = P_i + \frac{\nu}{1-\nu}(P_j + P_k), \quad g_j = g_k = 0. \tag{32}$$

From the obtained vector **g** it is seen that the localization mode corresponds to a split mode discontinuity.

To conclude it should be emphasized that the critical hardening modulus for localization (18) is obtained by finding the maximum over all the presented extremal solutions.

Examining the above cases it may be seen that when tensor **P** takes on the particular form (24), and excluding the special cases of infinite solutions, the band cannot form unless one of the components of its normal versor is null in the principal stress coordinate system. The above also holds in the case of the associative flow rule, as discussed below.

Case of associative plasticity

For associative plasticity, $\mathbf{P} = \mathbf{Q}$, case (i) is *a priori* excluded. Case (ii) reduces to:

$$n_i^2 = \frac{Q_i + \nu Q_k}{Q_i - Q_j}$$
$$n_j^2 = 1 - n_i^2, \quad n_k = 0. \tag{33}$$

When components (33) are internal to the inverval [0, 1], the corresponding extremum is:

$$H' = -2G(1+\nu)Q_k^2. \tag{34}$$

Vector $\mathbf{g}$ is finally obtained from (17) as:

$$g_i = (Q_i - Q_j)n_i$$
$$g_j = (Q_j - Q_i)n_j, \quad g_k = 0 \quad (j \text{ not summed}). \tag{35}$$

From the obtained vectors $\mathbf{n}$ and $\mathbf{g}$, eqns (15) yield:

$$[\![\dot{\varepsilon}_i]\!] = Q_i + \nu Q_k$$
$$[\![\dot{\varepsilon}_j]\!] = Q_j + \nu Q_k$$
$$[\![\dot{\varepsilon}_k]\!] = 0. \tag{36}$$

When the components (33) are admissible for at least two permutations of k between 1 and 3, the critical hardening modulus (18) is one of extrema (34). If two sets of components (33) are not admissible, the critical hardening modulus is also to be searched by examining case (iii), i.e.:

$$H' = -2G[(Q_j + \nu Q_k)^2/(1-\nu) + (1+\nu)Q_k^2] \tag{37}$$

corresponding to $n_i = 1, n_j = n_k = 0$. The localization mode is therefore a split discontinuity.

From eqns (34) and (37) it is easily seen that the hardening modulus, corresponding to strain localization, is never positive for associative plasticity. From eqn (34) it can also be noted that the normal vector to the band lies in the plane formed by the principal stress axes σ_i and σ_j, when the components (33) are admissible. It therefore appears that, when the minimum value of Q_k^2 corresponds to the direction of the minimum principal stress and the corresponding components of $\mathbf{n}$ are admissible, the normal to the band is orthogonal to the minimum principal stress direction. This may take place, as shown by Rudnicki and Rice (1975) for the Drucker–Prager yield surface, but not for the Huber–von Mises yield surface.

Plane strain and plane stress conditions

In the case of plane strain, as opposed to the general unconstrained case [eqns (25)–(37)], the position of the shear band is assumed to be *a priori* constrained; namely the normal to the band is assumed to lie in the plane of deformation. In such a case solutions (28)–(37) obtained for 3-D deformations in cases (ii) and (iii) hold true, assuming that direction k, normal to the plane of deformation, is a principal direction of stress. Case (i) is *a priori* excluded.

For plane stress the following circumstances have to be taken into account. First, following Thomas (1961), the solid is considered as two-dimensional. Therefore, it is assumed that the band reduces to a line and vectors $\mathbf{n}$ and $\mathbf{g}$, on which Maxwell compatibility conditions (14) are imposed, have two eigenvalues only. Moreover, due to the condition of the two-dimensional stress state, the elasticity tensor takes on a particular form. Thus, under the above conditions, the maximization problem (18) reduces to the following expression:

$$H^{l}_{cr} = \max_{\mathbf{n}} \{2G[2\mathbf{n}\cdot\mathbf{P}\cdot\mathbf{Q}\cdot\mathbf{n}-(\mathbf{n}\cdot\mathbf{P}\cdot\mathbf{n})(\mathbf{n}\cdot\mathbf{Q}\cdot\mathbf{n})-\mathbf{P}:\mathbf{Q}-\nu(\mathbf{n}\cdot\mathbf{Q}\cdot\mathbf{n}-\mathrm{tr}\,\mathbf{Q})(\mathbf{n}\cdot\mathbf{P}\cdot\mathbf{n}-\mathrm{tr}\,\mathbf{P})]\} \tag{38}$$

subject to $|\mathbf{n}| = 1$.

Vector $\mathbf{g}$ is expressed by:

$$\mathbf{g} = 2\mathbf{n}\cdot\mathbf{P}-(1+\nu)(\mathbf{n}\cdot\mathbf{P}\cdot\mathbf{n})\mathbf{n}+\nu(\mathrm{tr}\,\mathbf{P})\mathbf{n}. \tag{39}$$

In eqns (38) and (39) the index range is 1–2 and tensors **P** and **Q** now have only two eigenvectors.

We shall follow here the maximization procedure discussed in the 3-D case. However, the solutions of (38) may now be categorized into the following three cases:

(1) $Q_i \neq Q_j$ and $P_i \neq P_j$;
(2) $Q_i = Q_j$ and $P_i \neq P_j$ or $Q_i \neq Q_j$ and $P_i = P_j$;
(3) $Q_i = Q_j$ and $P_i = P_j$.

We now discuss them in detail.

(1) $Q_i \neq Q_j$ *and* $P_i \neq P_j$. Tensors **Q** and **P** do not have symmetries and the solution of (38) is:

$$H^{l}_{cr} = 2G(1+\nu)\max_{i=1,2}\left\{\frac{(P_1Q_2-P_2Q_1)^2}{4(P_1-P_2)(Q_1-Q_2)}, -P_iQ_i\right\} \tag{40}$$

where indices, not summed, denote components in the reference system of principal stresses. The two terms in brackets in (40) refer respectively to the case of the two non-zero components and of only one non-zero component of versor **n**.

If the first of the two terms in (40) gives the maximum H^{l}_{cr}, then the components of the versor **n** and of the vector **g** are:

$$n_1^2 = \frac{P_1}{2(P_1-P_2)}+\frac{Q_1}{2(Q_1-Q_2)}$$
$$n_2^2 = 1-n_1^2 \tag{41}$$

$$g_1 = \{2P_1-(1+\nu)[n_1^2(P_1-P_2)+P_2]+\nu(P_1+P_2)\}n_1$$
$$g_2 = \{2P_2-(1+\nu)[n_1^2(P_1-P_2)+P_2]+\nu(P_1+P_2)\}n_2. \tag{42}$$

If tensors **Q** and **P** in (41) are such that the values of components of versor **n** are external to the interval [0, 1], the corresponding value of the hardening modulus is to be rejected as non-admissible.

If the second term in H^{l}_{cr} maximizes (40), then the components of versor **n** and vector **g** are:

$$n_i = 1, \qquad n_j = 0 \tag{43}$$

$$g_i = P_i+\nu P_j, \quad g_j = 0. \tag{44}$$

Consequently, through (17), the corresponding localization mode is a split mode discontinuity.

The deformation rate jump inside the band may be obtained from expressions (15a) and (15c), together with the value of versor **n** and vector **g**.

(2) $Q_i = Q_j$ *and* $P_i \neq P_j$ *or* $Q_i \neq Q_j$ *and* $P_i = P_j$. In this case the only possible solution occurs when the band is orthogonal to a principal stress direction. Therefore the critical hardening modulus is given by the second term on the right-hand side of (40).

(3) $Q_i = Q_j$ *and* $P_i = P_j$. In this case all the inclinations of the band are possible. The critical hardening modulus is given by the second expression on the right-hand side of (40). The localization mode is a split mode discontinuity.

Associative plane stress plasticity

For associative plasticity, eqns (40)–(44) reduce to:

$$H^I_{cr} = \max_{i=1,2}[0, -2G(1+\nu)Q_i^2]. \tag{45}$$

Therefore the maximum corresponds to a null hardening modulus, if the corresponding band inclination is admissible:

$$n_1^2 = \frac{Q_1}{Q_1 - Q_2}, \quad n_2^2 = 1 - n_1^2. \tag{46}$$

Vector **g** results:

$$g_1 = (Q_1 - Q_2)n_1, \quad g_2 = -(Q_1 - Q_2)n_2. \tag{47}$$

In this case eqns (15a) and (15c) yield:

$$[\![\dot{\varepsilon}_1]\!] = Q_1, \quad [\![\dot{\varepsilon}_2]\!] = Q_2. \tag{48}$$

If tensor **Q** is such as to give components (46) (including the case $Q_1 = Q_2$) external to the interval [0, 1], then the critical hardening modulus is given by the second expression in brackets in (45). This corresponds to the band inclination defined by (43) and to the corresponding vector **g** given by (44) (where **P** is substituted by **Q**). The localization mode is a split mode discontinuity.

Comments: perfect plasticity, associative flow rule

In the case of plane strain, we observe the following.

(i) At collapse, only plastic strain rate occurs. Thus Q_k must vanish at collapse and therefore, from eqn (34), $H^I_{cr} = 0$. It is then concluded that, according to an incremental theory such as that described through (7)–(11), collapse may be attained with the formation of discontinuity bands.

(ii) In the case of zero volumetric plastic strain, the additional condition $Q_i = -Q_j$ must hold at collapse. The resulting shear band always has an admissible inclination equal to 45° with respect to the principal maximum stress direction, as seen from (33).

In the case of plane stress, we observe the following.

(iii) Also in this case the classical result for thin sheet uniaxial tension of Thomas (1961) is recovered. In fact, for the Huber–von Mises yield criterion the shear band is inclined at 54.73° to the direction of principal tensile stress, as seen from eqn (46).

To conclude, it should be noted that the expressions (25)–(37) for the localization hardening modulus H^I_{cr} and inclination of the band are a generalization, to any form of the yield surface and of the non-associative flow rule, of the result obtained earlier by Rudnicki and Rice (1975) for Drucker–Prager and Huber–von Mises yield conditions with a particular non-associative flow rule. The expressions obtained are given in an explicit (decoupled) form and the critical hardening modulus and the localization mode may be found without numerical procedures. Equations (38)–(48) furnish in turn a specialization of the critical hardening modulus for localization and band inclination for cases of plane strain and plane stress.

4. SECOND ORDER WORK CRITERIA FOR 3-D, PLANE STRAIN AND PLANE STRESS CASES

The threshold corresponding to zero second order work (Mróz, 1963), as given for general unconstrained kinematics (3-D) by Maier and Hueckel (1977, 1979) in terms of a critical hardening modulus, is rewritten in eqn (12).

In the present section this criterion is specialized to plane strain and plane stress conditions.† As may be seen from eqn (13) the deformation rate for which the zero second order work is attained is, in general, three-dimensional and thus may violate the plane strain condition. Therefore, an additional constraint, namely that of vanishing of the prescribed components of the strain rate must be imposed on (4).

In plane strain, the second order work is produced only by the in-plane deformation rate components, i.e.:

$$\dot{\varepsilon}_{33} = \dot{\varepsilon}_{13} = \dot{\varepsilon}_{32} = 0, \quad \dot{\varepsilon}_{11} \neq 0, \quad \dot{\varepsilon}_{22} \neq 0, \quad \dot{\varepsilon}_{21} = \dot{\varepsilon}_{21} \neq 0 \tag{49}$$

and thus:

$$W = \tfrac{1}{2}\dot{\boldsymbol{\varepsilon}} : \mathbf{D}' : \dot{\boldsymbol{\varepsilon}}, \quad (\dot{\boldsymbol{\varepsilon}} \in \mathbb{R}^2 \otimes \mathbb{R}^2) \tag{50}$$

where $\mathbf{D}'$ is the elastoplastic tensor in plane strain:

$$\mathbf{D}' = \mathbf{E}' - \frac{\mathbf{M} \otimes \mathbf{N}}{H - H_0}. \tag{51}$$

In eqn (51), $\mathbf{E}'$ is the isotropic elastic tensor in plane strain and the scalar H_0 is defined as:

$$H_0 = -P_{ij} E_{ijhk} Q_{hk} \quad (i,j,h,k = 1,2,3) \tag{52}$$

or:

$$H_0 = \lambda(\operatorname{tr} \mathbf{P})(\operatorname{tr} \mathbf{Q}) + 2G\mathbf{P} : \mathbf{Q} \tag{53}$$

where λ is Lamé's constant defined as $\lambda = 2G/(1-2\nu)$.

Finally, in eqn (51), tensors $\mathbf{N}$ and $\mathbf{M} \in \mathbb{R}^2 \otimes \mathbb{R}^2$ are defined as:

$$N_{hk} = Q_{ij} E_{ijhk}, \quad M_{hk} = P_{ij} E_{ijhk} \tag{54}$$

where free indices range between 1 and 2, but the dummy indices range is 1–3.

For every elastic incremental process ($\dot{\Lambda} = 0$), second order work is always positive, because $\mathbf{D}'$ reduces to $\mathbf{E}'$, which is positive definite. For the elastoplastic process, loss of positive definiteness of plane strain tensor $\mathbf{D}'$ may be determined through a minimization of the second order work (following Maier and Hueckel, 1979):

$$\min_{\dot{\varepsilon}} \tfrac{1}{2}\{\dot{\boldsymbol{\varepsilon}} : \mathbf{E}' : \dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}} : \mathbf{M}\} \tag{55}$$

subject to:

† It may be shown that in plane strain and plane stress loss of second order work positive definiteness for the comparison solid "in loading" occurs at the same critical value of hardening modulus as for the best chosen Raniecki's comparison solid. Therefore, only the comparison solid "in loading" needs to be considered.

$$\dot{\boldsymbol{\varepsilon}}:\mathbf{N} = H - H_0. \tag{56}$$

$\dot{\Lambda} = 1$ has been assumed in eqn (55), because only the sign of second order work matters for positive definiteness of $\mathbf{D}'$.

The optimal tensor $\dot{\boldsymbol{\varepsilon}}^0$ that minimizes (55) is:

$$\dot{\boldsymbol{\varepsilon}}^0 = \tfrac{1}{2}\mathbf{C}':\mathbf{M} - \omega\mathbf{C}':\mathbf{N} \tag{57}$$

where:

$$\omega = (\tfrac{1}{2}\mathbf{N}:\mathbf{C}':\mathbf{M} - H + H_0)(\mathbf{N}:\mathbf{C}':\mathbf{N})^{-1} \tag{58}$$

is a Lagrangian multiplier and $\mathbf{C}'$ is the plane strain elastic compliance tensor:

$$C'_{ijkh} = -\frac{\nu}{2G}\delta_{ij}\delta_{kh} + \frac{1}{4G}(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}) \quad (i,j,k,h = 1,2) \tag{59}$$

where δ_{ij} is the Kronecker delta symbol.

Zero second order work (50) produced by the strain rate (57) occurs at the value of the hardening modulus H^0_{cr} found as:

$$H^0_{cr} = \tfrac{1}{2}[\sqrt{(\mathbf{M}:\mathbf{C}':\mathbf{M})(\mathbf{N}:\mathbf{C}':\mathbf{N})} + \mathbf{N}:\mathbf{C}':\mathbf{M} + 2H_0]. \tag{60}$$

The quadratic forms in eqn (60) may be specialized via (59) and (54) as:

$$\begin{aligned}
\mathbf{M}:\mathbf{C}':\mathbf{M} &= \lambda(P_{ii})^2 + 2GP_{ij}P_{ij} - 4GP_{i3}P_{i3} + 2G(1-\nu)P_{33}^2 \\
\mathbf{N}:\mathbf{C}':\mathbf{N} &= \lambda(Q_{ii})^2 + 2GQ_{ij}Q_{ij} - 4GQ_{i3}Q_{i3} + 2G(1-\nu)Q_{33}^2 \\
\mathbf{N}:\mathbf{C}':\mathbf{M} &= \lambda(P_{ii})(Q_{jj}) + 2GP_{ij}Q_{ij} - 4GP_{i3}Q_{i3} + 2G(1-\nu)P_{33}Q_{33} \\
&\qquad (i,j = 1,2,3).
\end{aligned} \tag{61}$$

The strain rate (57) that makes the second order work equal to zero is found to be:

$$\dot{\boldsymbol{\varepsilon}}^0 = \alpha[\mathbf{C}':\mathbf{N}\sqrt{(\mathbf{M}:\mathbf{C}':\mathbf{M})/(\mathbf{N}:\mathbf{C}':\mathbf{N})} + \mathbf{C}':\mathbf{M}], \quad \forall\alpha\in\mathbb{R}-\{0\}. \tag{62}$$

The case of plane stress does not require particular calculations. In fact, the deformation for the plane stress case is not constrained and eqns (12) and (13) still hold. The only restriction is that now the fourth order elastic tensor $\mathbf{E}$ is that of plane stress and $\mathbf{P}$ and $\mathbf{Q}$ have only two principal components. Thus eqns (12) and (13) are replaced respectively by:

$$H^0_{cr} = \tfrac{1}{2}[\sqrt{[\vartheta(\mathrm{tr}\,\mathbf{Q})^2 + 2G\mathbf{Q}:\mathbf{Q}][\vartheta(\mathrm{tr}\,\mathbf{P})^2 + 2G\mathbf{P}:\mathbf{P}]} - [\vartheta(\mathrm{tr}\,\mathbf{P})(\mathrm{tr}\,\mathbf{Q}) + 2G\mathbf{P}:\mathbf{Q}]] \tag{63}$$

$$\dot{\boldsymbol{\varepsilon}}^0 = \alpha[\sqrt{[\vartheta(\mathrm{tr}\,\mathbf{Q})^2 + 2G\mathbf{Q}:\mathbf{Q}]}\,\mathbf{P} + \sqrt{[\vartheta(\mathrm{tr}\,\mathbf{P})^2 + 2G\mathbf{P}:\mathbf{P}]}\,\mathbf{Q}], \quad \forall\alpha\in\mathbb{R}-\{0\} \tag{64}$$

where $\vartheta = 2\nu G/(1-\nu)$.

Plane strain and plane stress associative plasticity

In the case of plane strain associative plasticity, eqns (60) and (62) reduce to:

$$H^0_{cr} = 2G[-2Q_{i3}Q_{i3} + (1-\nu)Q_{33}^2] \quad (i = 1,2,3) \tag{65}$$

$$\dot{\boldsymbol{\varepsilon}}^0 = \mathbf{C}':\mathbf{N} \quad \text{or} \quad \dot{\varepsilon}^0_{ij} = \nu Q_{33}\delta_{ij} + Q_{ij} \quad (i,j = 1,2). \tag{66}$$

Note that, if eqn (65) is expressed in the principal stress reference system, it coincides with

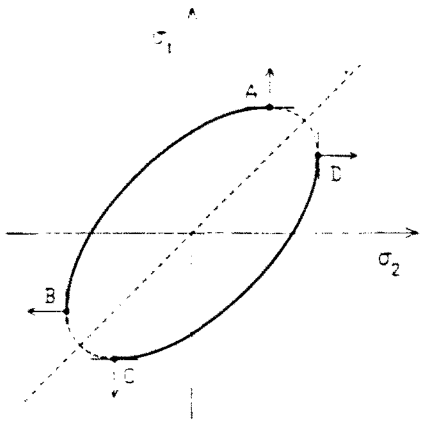


Fig. 2. Ranges of coincidence of criteria for localization and for second order work in the case of the Huber–von Mises condition in plane stress.

eqn (34) in which $k = 3$ is assumed as the direction normal to the plane of deformation. A coincidence is also observed between eqns (66) and (36). Thus it is concluded that the formation of shear bands occurs at the critical modulus for the loss of second order work positive definiteness.

For plane stress associative plasticity, eqns (63) and (64) reduce to:

$$H''_{cr} = 0 \tag{67}$$

$$\dot{\varepsilon}^{0} = \alpha \mathbf{Q}, \quad \forall \alpha \in \mathbb{R} - \{0\}. \tag{68}$$

Thus, also for plane stress associative plasticity the critical hardening moduli for shear band formation and for loss of second order work positive definiteness coincide [see eqns (45—first term) and (48), eqns (67) and (68)].

The above coincidences have the following implications. The vanishing of the second order work in plane strain and plane stress associative plasticity occurs simultaneously with strain localization in cases when the solutions (34) and (45—first term) are admissible, that is when a shear band forms. The localization and second order work criteria also coincide when a split mode is associated with an analytical maximum. For instance, coincidence between the two criteria always occurs in the case of plane strain for the Huber–von Mises yield function, if the intermediate principal stress is orthogonal to the plane of deformation. However, for the plane stress the coincidence occurs only on the portions AB and CD of the yield curve (including points ABCD; Fig. 2).

The above means that in plane, associative elastoplasticity, shear band formation always coincides with the loss of second order work positive definiteness, while this may not be the case when a split mode localization takes place. In other words, during a loading process under homogeneous conditions only two possibilities may arise at the instant of localization, excluding reversal of the sign of the second order work:

(i) A shear band forms. Then the localization criterion coincides with the criterion of the second order work and any loss of uniqueness is excluded prior to band formation.

(ii) A split mode discontinuity occurs. Then the threshold of the second order work has been attained before, or at the instant of, split mode localization.

In the case examined by Hill and Hutchinson (1975), corresponding to large displacement gradients in plane strain associative plasticity, an infinite number of diffuse bifurcation modes can precede localization into shear bands. However, this loss of uniqueness before localization may be shown to disappear when the effects due to large displacement gradients are neglected.

The appearance of diffuse bifurcation modes prior to shear band formation is possible in general 3-D small deformation problems and/or for non-associative plasticity. Therefore a gap exists between the thresholds for second order work and for localization. Hence, in the above sense, alternative bifurcation modes may occur at hardening moduli between those corresponding to these two events.

5. CONCLUSIONS

The results obtained in the preceding sections pertain to the relationships between thresholds for uniqueness, bifurcation and stability as presented in Section 2.

The following conclusions have been reached concerning associative and non-associative elastoplasticity in 3-D, plane strain and plane stress. The first group of results obtained was inspired by the possibility for generalization of some of the conclusions regarding the localization criterion obtained by Rudnicki and Rice (1975) for the Drucker–Prager yield condition (Huber–von Mises as a particular case) with the non-associative flow rule.

(1) Starting from the condition of localization given by Rice (1976), an explicit solution for the critical hardening modulus is found. This result concerns any smooth yield surface and any direction of plastic flow. The existence of an explicit solution makes it possible to directly inspect all possible localization modes.

(2) It is confirmed that the critical hardening modulus for strain localization is never positive for associative plasticity (Rice, 1976).

(3) The normal to the band (excluding the cases of infinite solutions) is perpendicular to a principal direction of stress. This is always true for associative plasticity and for a class of non-associative flow rules (e.g. Jenike and Shield, 1959), but may not be the case for other non-associative plastic models. Moreover, the normal to the shear band may be orthogonal to the direction of the minimum principal stress.

(4) A specialization of the criterion of second order work positiveness to plane strain and plane stress shows that the critical hardening modulus is never positive in plane strain and is equal to zero in plane stress for associative plasticity.

(5) A specialization of the criterion for strain localization to plane stress and plane strain shows that, for the associative flow rule, the critical hardening modulus for strain localization into shear bands coincides with the critical hardening modulus at loss of positive definiteness of second order work. In contrast, localization into splitting modes may not occur at the threshold of second order work. When the conditions are met for the coincidence of the criteria of second order work and localization:

non-uniqueness cannot occur in a boundary value problem before localization of deformation into a planar band;

loss of positiveness of second order work, loss of strong ellipticity and loss of ellipticity (i.e. strain localization and vanishing of the speed of acceleration waves) are equivalent criteria.

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